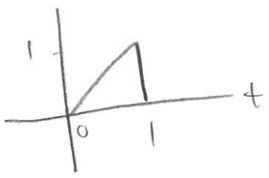
1. $4, 2 - 2$
2. $4, 2 - 3$
3. $4, 2 - 4$
4. $4.5 - 2$

Spoiler Alert! Answers on next page.

1.a



$$x(t) = t(u(t) - u(t-1))$$

I'm using unit steps instead of a box because $u(t)$ "fits" with Laplace techniques better. Clearly it has a simpler transform than $\Pi(t)$, it's causal, and it just how I learned it.

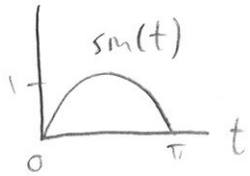
$$t\,u(t) \xrightarrow{\mathcal{L}} \frac{1}{s^2} \qquad u(t-1) \xrightarrow{\mathcal{L}} \frac{e^{-s}}{s}$$

$$(t-1)u(t-1) \xrightarrow{\mathcal{L}} \frac{e^{-s}}{s^2}$$

$$x(t) = t\,u(t) - t\,u(t-1) = t\,u(t) - (t-1)u(t-1) - u(t-1)$$

$$\boxed{X(s) = \frac{1}{s^2}\left(1 - e^{-s} - se^{-s}\right)}$$

1.b



$$x(t) = \sin(t)(u(t) - u(t-\pi))$$

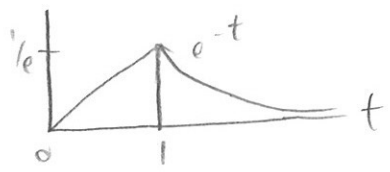
$$\sin(t)\,u(t) \xrightarrow{\mathcal{L}} \frac{1}{s^2+1} \qquad \sin(t-\pi) = -\sin(t)$$

$$\sin(t-\pi)u(t-\pi) \xrightarrow{\mathcal{L}} \frac{e^{-\pi s}}{s^2+1}$$

$$x(t) = \sin(t)\,u(t) + \sin(t-\pi)u(t-\pi) \xrightarrow{\mathcal{L}}$$

$$\boxed{X(s) = \frac{1 - e^{-\pi s}}{s^2+1}}$$

1.c



$$x(t) = e^{-1} t\,(u(t)-u(t-1)) + e^{-t}u(t-1)$$

$$= e^{-1}\underbrace{t\,(u(t)-u(t-1))}_{\text{from part (a)}} + e^{-1}e^{-(t-1)}u(t-1)$$

$$\boxed{X(s) = e^{-1}\left(\frac{1}{s^2}\left(1-e^{-s}-se^{-s}\right) + \frac{e^{-s}}{s+1}\right)}$$

$$e^{-t}u(t) \xrightarrow{\mathcal{L}} \frac{1}{s+1}$$

$$e^{-(t-1)}u(t-1) \xrightarrow{\mathcal{L}} \frac{e^{-s}}{s+1}$$

2.a

$$X(s) = \frac{(2s+5)e^{-2s}}{s^2+5s+6} = \frac{(2s+5)e^{-2s}}{(s+2)(s+3)} = e^{-2s}\left(\frac{A}{s+2} + \frac{B}{s+3}\right)$$

$$A = \lim_{s\to -2} \frac{2s+5}{s+3} = \frac{-4+5}{1} = +1$$

$$B = \lim_{s\to -3} \frac{2s+5}{s+2} = \frac{-6+5}{-1} = 1$$

$$X(s) = e^{-2s}\left(\frac{1}{s+3} + \frac{1}{s+2}\right) \xleftrightarrow{\mathcal{L}^{-1}} \boxed{\left(e^{-3(t-2)} + e^{-2(t-2)}\right)u(t-2)}$$

2.b

$$X(s) = \frac{se^{-3s}+2}{s^2+2s+2} = \frac{2}{(s-p)(s-p^*)} + \frac{se^{-3s}}{(s-p)(s-p^*)} \qquad p = -(1+j)$$

trick: use the frequency shift rule:

$$s^2+2s+2 = (s^2+2s+1)+1 = (s+1)^2+1$$ (could also just use partial fractions)

$$\frac{2}{(s+1)^2+1} \xrightarrow{\mathcal{L}^{-1}} 2e^{-t}\sin(t)\,u(t)$$

$$\frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1} \xrightarrow{\mathcal{L}^{-1}} e^{-t}(\cos(t)-\sin(t))u(t)$$

$$\boxed{x(t) = 2e^{-t}\sin(t)\,u(t) + e^{-(t-3)}(\cos(t-3)-\sin(t-3))u(t-3}$$

2.c $X(s) = \dfrac{e^{-(s-1)} + 3}{s^2 - 2s + 5}$

$s^2 - 2s + 5 = (s^2 - 2s + 1) + 4 = (s-1)^2 + 2^2$

$\dfrac{3}{(s-1)^2 + 2^2} = \dfrac{3}{2} \dfrac{2}{(s-1)^2 + 2^2} \xrightarrow{\mathcal{L}^{-1}} \dfrac{3}{2} e^{t} \sin(2t)\, u(t)$

$\dfrac{1}{s^2 + 2^2} = \dfrac{1}{2} \dfrac{2}{s^2 + 2^2} \xrightarrow{\mathcal{L}^{-1}} \dfrac{1}{2} \sin(2t)\, u(t)$

$\dfrac{1}{2} \dfrac{2e^{-s}}{s^2 + 2^2} \xrightarrow{\mathcal{L}^{-1}} \dfrac{1}{2} \sin(2(t-1))\, u(t-1)$

$\dfrac{1}{2} \dfrac{2e^{-(s-1)}}{(s-1)^2 + 2^2} \xrightarrow{\mathcal{L}^{-1}} \dfrac{1}{2} e^{t} \sin(2(t-1))\, u(t-1)$

The frequency shift is applied after the e^{-s}. That is why the e^{t} is not time shifted. Understand this.

$$x(t) = \left(\frac{3}{2} \sin(2t)\, u(t) + \frac{1}{2} \sin(2(t-1))\, u(t-1) \right) e^{t}$$

2.d $X(s) = \dfrac{1 + e^{-s} + e^{-2s}}{s^2 + 3s + 2} = \dfrac{1 + e^{-s} + e^{-2s}}{(s+1)(s+2)} = (1 + e^{-s} + e^{-2s}) \left(\dfrac{A}{s+1} + \dfrac{B}{s+2} \right)$

$A = \lim_{s \to -1} \dfrac{1}{s+2} = 1 \qquad B = \lim_{s \to -2} \dfrac{1}{s+1} = -1$

$$g(t) = \mathcal{L}^{-1}\left\{ \frac{1}{s+1} - \frac{1}{s+2} \right\} = \left(e^{-t} - e^{-2t} \right) u(t)$$

$$x(t) = g(t) + g(t-1) + g(t-2)$$

3.a First of all, the only periodic, causal signal is 0 (the zero signal), causality ruins the periodicity. Still, it still fits with the notion of a repeating signal, and that's what I'll call it: causal repeating signal.

This is extremely useful in analyzing transient behavior (of signals and systems, not so much migrating hobos.)

$x(t)$ is of finite duration, and is causal.

$$x(t) = x(t)\left(u(t) - u(t-T_0)\right)$$

$$g(t) = \sum_{n=0}^{\infty} x(t-nT_0) \quad \leftarrow \text{partial Poisson sum.}$$

$$\mathcal{L}\{g(t)\} = \int_0^{\infty} \sum_{n=0}^{\infty} x(t-nT_0)\, e^{-st}\, dt$$

$$= \sum_{n=0}^{\infty} \int_0^{\infty} x(t-nT_0)\, e^{-st}\, dt$$

$$= \sum_{n=0}^{\infty} \mathcal{L}\{x(t-nT_0)\} = \sum_{n=0}^{\infty} X(s)\, e^{-nT_0 s}$$

$$= X(s) \sum_{n=0}^{\infty} e^{-nT_0 s}$$

geometric series $\sum_{n=0}^{\infty} \left(e^{-sT_0}\right)^n = \frac{1}{1-e^{-sT_0}}$ if $\left|e^{-sT_0}\right| = e^{-\sigma T_0} < 1 \Rightarrow \sigma T_0 > 0 \Rightarrow \sigma > 0$

$$G(s) = \frac{X(s)}{1-e^{-sT_0}}, \quad \mathrm{Re}(s) > 0$$

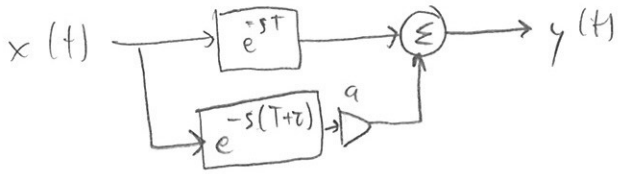
(since $x(t)$ is of finite duration, it's RoC is $\mathbb{C}$ for most practical signals).

3.b $x(t) = u(t) - u(t-2) \overset{\mathcal{L}}{\Rightarrow} X(s) = \frac{1-e^{-2s}}{s}$

$T_0 = 8$

$$\boxed{G(s) = \frac{1-e^{-2s}}{s\left(1-e^{-8s}\right)}}$$

4.



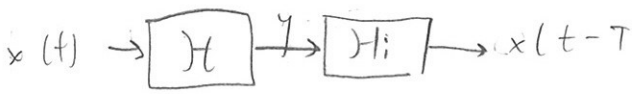
$y(t) = x(t-T) + a\,x(t-(T+\tau))$ This is LTI. You may want to check for yourself.

$Y(s) = e^{-sT}X(s) + a e^{-s(T+\tau)}X(s)$

$= e^{-sT}X(s)\left(1 + a e^{-s\tau}\right)$

$H(s) = e^{-sT}\left(1 + a e^{-s\tau}\right)$

As the problem says, we only care about the τ piece. The delay by T is ok.



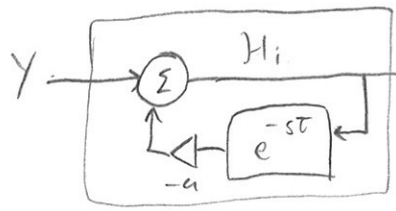
$H_i(s)H(s) = e^{-sT_1} \rightarrow H_i(s) = \frac{e^{-sT_1}}{H(s)} = \frac{e^{-sT_1}}{e^{-sT}(1+a e^{-s\tau})} = \frac{e^{-s(T_1-T)}}{1+a e^{-s\tau}}$

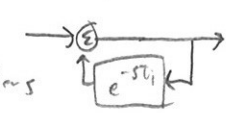
$X_i(s) = H_i(s)Y(s) \rightarrow \left(1 + a e^{-s\tau}\right)X_i(s) = e^{-s(T_1-T)}Y(s) = Y(s)$ (set T_1 to T)

$X_i(s)$: output; $Y(s)$: input

$X_i = Y - a e^{-s\tau}X_i \rightarrow$

$x_i(t) = y(t) - a\,x_i(t-\tau)$



The system H_i can be built. It is causal (this should be obvious).

The hard part is figuring out what a and τ are. A simple way to estimate them is with a correlator, which basically takes the system for several different τ_i and picks the best one. Usually echo cancellers are done in software with DSPs these days.